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Functional inequalities for the incomplete gamma function

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ABSTRACT

We present several inequalities for

$$f_a(x)=\frac{\Gamma(a,x)}{\Gamma(a,0)}\quad (a>0,\ x\geqslant 0),$$

where $\Gamma(a,x)$ is the incomplete gamma function. One of our theorems states that the inequalities

$$f_a\big(S_p(x_1,\ldots,x_n)\big)\leqslant f_a(x_1)\cdots f_a(x_n)\leqslant f_a\big(S_q(x_1,\ldots,x_n)\big)\quad (p,q>0)$$

hold for all nonnegative real numbers $x_1,\ldots,x_n$ $(n\geqslant 2)$ if and only if $p\leqslant\min(a,1)$ and $q\geqslant\max(a,1)$. Here, $S_t(x_1,\ldots,x_n)$ denotes the power sum of order t. This extends and complements a result published by Ismail and Laforgia in 2006.

1. Introduction

The incomplete gamma function, defined for real numbers $a>0$ and $x\geqslant 0$ by

$$\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt,$$

has numerous applications in statistics, probability theory, and other fields. The most important properties of this function are collected, for example, in [1, Chapter 6]. Many information on the incomplete gamma function with interesting historical comments and a detailed list of references can be found in [11].

Throughout this paper, we denote by f_a the 'normalized' function

$$f_a(x)=\frac{\Gamma(a,x)}{\Gamma(a,0)}.$$

The function f_{a+1} is the unique solution of the linear differential equation

$$y'+y=f_a(x),\qquad y(0)=1.$$

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In 2006, Ismail and Laforgia [14] presented remarkable functional inequalities for f_a. They proved for $x, y \geqslant 0$:

$$f_a(x+y) \leqslant f_a(x)f_a(y) \quad (a>1) \quad \text{and} \quad f_a(x)f_a(y) \leqslant f_a(x+y) \quad (0<a<1). \tag{1}$$

We denote by $S_t(x_1, \dots, x_n)$ the power sum of order t, that is,

$$S_t = \left(x_1^t + \cdots + x_n^t\right)^{1/t} \quad (t \neq 0).$$

Using this notation (1) can be written as

$$f_a\big(S_1(x,y)\big) \leqslant f_a(x)f_a(y) \quad (a>1) \quad \text{and} \quad f_a(x)f_a(y) \leqslant f_a\big(S_1(x,y)\big) \quad (0<a<1), \tag{2}$$

respectively. In the next section we generalize (2). We provide all parameters p and q such that the double-inequality

$$f_a\big(S_p(x_1,\dots,x_n)\big) \leqslant f_a(x_1)\cdots f_a(x_n) \leqslant f_a\big(S_q(x_1,\dots,x_n)\big) \tag{3}$$

holds for all $x_1, \dots, x_n \geqslant 0$. Furthermore, we offer some mean-value inequalities. The power mean of order t is defined by

$$M_t(x_1,\dots,x_n) = \left(\frac{x_1^t + \cdots + x_n^t}{n}\right)^{1/t} \quad (t \neq 0), \qquad M_0(x_1,\dots,x_n) = (x_1 \cdots x_n)^{1/n},$$
$$M_{-\infty}(x_1,\dots,x_n) = \min(x_1,\dots,x_n), \qquad M_{\infty}(x_1,\dots,x_n) = \max(x_1,\dots,x_n).$$

A detailed study of power means is given in [10, Chapter III]. We determine all parameters r, u, α and s, v, β such that we have for all $x_1, \dots, x_n > 0$:

$$f_a\big(M_r(x_1,\dots,x_n)\big) \leqslant \frac{f_a(x_1) + \cdots + f_a(x_n)}{n} \leqslant f_a\big(M_s(x_1,\dots,x_n)\big),$$
$$f_a\big(M_u(x_1,\dots,x_n)\big) \leqslant \big(f_a(x_1)\cdots f_a(x_n)\big)^{1/n} \leqslant f_a\big(M_v(x_1,\dots,x_n)\big),$$

and

$$f_a\big(M_\alpha(x_1,\dots,x_n)\big) \leqslant \frac{n}{1/f_a(x_1) + \cdots + 1/f_a(x_n)} \leqslant f_a\big(M_\beta(x_1,\dots,x_n)\big).$$

In 1993, motivated by the Turán-type inequality

$$\big(1 - f_a(x)\big)\big(1 - f_{a+2}(x)\big) < \big(1 - f_{a+1}(x)\big)^2,$$

Merkle [17] conjectured that for every $x > 0$ the function $a \mapsto 1 - f_a(x)$ is log-concave on $(0, \infty)$. A proof of this conjecture can be found in [2]. It is natural to ask whether $a \mapsto f_a(x)$ $(x > 0)$ is also log-concave on $(0, \infty)$. In the final part of Section 2, we give an affirmative answer to this question.

In Section 3, we present several additional results. Among others, we provide all parameters b, c, such that $x \mapsto [f_a(x^b)]^c$ is subadditive on $[0, \infty)$ and we show that f_a is completely monotonic on $[0, \infty)$ if and only if $a \in (0, 1]$.

2. Inequalities

First, we offer convexity and concavity properties of functions, which are defined in terms of f_a.

Lemma 1. *Let*

$$u_a(x) = f_a\big(x^{1/a}\big), \qquad v_a(x) = \log f_a(x), \qquad w_a(x) = \log f_a\big(x^{1/a}\big), \qquad z_a(x) = \log f_a\big(e^x\big).$$

(i) *If $a > 0$, then u_a is strictly convex on $[0, \infty)$.*
(ii) *If $0 < a < 1$, then v_a is strictly convex on $[0, \infty)$ and w_a is strictly concave on $[0, \infty)$.*
(iii) *If $a > 1$, then v_a is strictly concave on $[0, \infty)$ and w_a is strictly convex on $[0, \infty)$.*
(iv) *If $a > 0$, then z_a is strictly concave on $\mathbf{R}$.*

Proof. Let $x > 0$. We obtain for $a > 0$:

$$u_a''(x) = \frac{e^{-x^{1/a}} x^{-1+1/a}}{a^2 \Gamma(a, 0)} > 0.$$

By differentiation we get

$$v_a''(x) = -\frac{e^{-x} x^{a-1}}{x \Gamma(a,x)^2} P_a(x), \tag{4}$$

where

$$P_a(x) = \Gamma(a,x)(a-1-x) + e^{-x}x^a.$$

Case 1. $0 < a < 1$.

We define

$$Q_a(x) = \frac{P_a(x)}{a-1-x} = \Gamma(a,x) + \frac{e^{-x}x^a}{a-1-x}. \tag{5}$$

Then we have

$$Q_a'(x) = \frac{(a-1)e^{-x}x^{a-1}}{(a-1-x)^2}. \tag{6}$$

This leads to

$$Q_a'(x) < 0 \quad \text{and} \quad Q_a(x) > \lim_{t\to\infty} Q_a(t) = 0. \tag{7}$$

From (5) and (7) we get $P_a(x) < 0$, so that (4) implies that v_a'' is positive on $(0,\infty)$.

Case 2. $a > 1$.

If $x \leqslant a-1$, then $P_a(x) > 0$. Let $x > a-1$. Applying (6) leads to

$$Q_a'(x) > 0 \quad \text{and} \quad Q_a(x) < \lim_{t\to\infty} Q_a(t) = 0.$$

Hence, $P_a(x) > 0$. Using (4) gives $v_a''(x) < 0$ for $x > 0$.

We have

$$w_a''(x) = \frac{e^{-z}z^{a+1}}{a^2x^2\Gamma(a,z)^2} R_a(z), \tag{8}$$

where

$$R_a(t) = \Gamma(a,t) - e^{-t}t^{a-1} \quad \text{and} \quad z = x^{1/a}.$$

Differentiation gives

$$R_a'(t) = (1-a)e^{-t}t^{a-2}.$$

Hence, we obtain for $t > 0$:

$$R_a(t) < \lim_{s\to\infty} R_a(s) = 0, \quad \text{if } 0 < a < 1, \tag{9}$$

and

$$R_a(t) > \lim_{s\to\infty} R_a(s) = 0, \quad \text{if } a > 1. \tag{10}$$

Combining (8) with (9) and (10), respectively, we conclude that $w_a''(x) < 0$, if $0 < a < 1$ and that $w_a''(x) > 0$, if $a > 1$.

We have

$$z_a''(x) = -\frac{e^{-y}y^a}{\Gamma(a,y)^2} D_a(y),$$

where

$$D_a(y) = (a-y)\Gamma(a,y) + e^{-y}y^a \quad \text{and} \quad y = e^x.$$

If $0 < y \leqslant a$, then $D_a(y) > 0$. Let $y > a$ and

$$E_a(y) = \frac{D_a(y)}{a-y} = \Gamma(a,y) + \frac{e^{-y}y^a}{a-y}.$$

Since

$$E_a'(y) = \frac{e^{-y} y^a}{(y-a)^2} > 0,$$

we obtain

$$E_a(y) < \lim_{t\to\infty} E_a(t) = 0.$$

This implies $D_a(y) > 0$. Thus, $z_a''(x) < 0$. □

Moreover, we need the following inequality, which is due to Petrović [19, p. 22].

Lemma 2. *If F is convex on $[0,\infty)$, then we have for $x_1,\dots,x_n \geqslant 0$:*

$$F(x_1) + \cdots + F(x_n) \leqslant F(x_1 + \cdots + x_n) + (n-1)F(0).$$

If F is concave on $[0,\infty)$, then the reversed inequality holds.

Our first theorem extends and complements (2).

Theorem 1. *Let a be a positive real number. The inequalities*

$$f_a\bigl(S_p(x_1,\dots,x_n)\bigr) \leqslant f_a(x_1)\cdots f_a(x_n) \leqslant f_a\bigl(S_q(x_1,\dots,x_n)\bigr) \quad (p,q>0) \tag{11}$$

hold for all nonnegative real numbers $x_1,\dots,x_n$ ($n \geqslant 2$) if and only if

$$p \leqslant \min(a,1) \quad \textit{and} \quad q \geqslant \max(a,1). \tag{12}$$

Proof. Since $t \mapsto S_t(x_1,\dots,x_n)$ is decreasing on $(0,\infty)$ (see [13, p. 28]) and

$$f_a'(x) = -\frac{e^{-x}x^{a-1}}{\Gamma(a,0)} < 0,$$

we conclude that the function

$$t \mapsto f_a\bigl(S_t(x_1,\dots,x_n)\bigr)$$

is increasing on $(0,\infty)$. Therefore, it suffices to establish (11) for $p = \min(a,1)$ and $q = \max(a,1)$.

We apply Lemma 1 (ii), (iii) and Lemma 2. If $0 < a < 1$, then we obtain

$$w_a\bigl(x_1^a + \cdots + x_n^a\bigr) \leqslant w_a\bigl(x_1^a\bigr) + \cdots + w_a\bigl(x_n^a\bigr) = v_a(x_1) + \cdots + v_a(x_n) \leqslant v_a(x_1 + \cdots + x_n). \tag{13}$$

If $a > 1$, then we get (13) with "$\geqslant$" instead of "$\leqslant$". And, if $a = 1$, then (13) holds with "$=$" instead of "$\leqslant$".

It remains to show that (11) implies (12). We set $x_1 = x_2 = x$ and $x_3 = \cdots = x_n = 0$. Then we have

$$f_a\bigl(2^{1/p}x\bigr) \leqslant f_a(x)^2 \leqslant f_a\bigl(2^{1/q}x\bigr) \quad (x>0). \tag{14}$$

Let $1 < c \neq 2$. Hospital's rule gives

$$\lim_{x\to\infty} \frac{f_a(cx)}{f_a(x)^2} = \lim_{x\to\infty} \frac{(c-1)c^a\Gamma(a,0)e^{(2-c)x}}{2x^{a-1}} = \begin{cases} \infty, & \text{if } 1<c<2, \\ 0, & \text{if } c>2. \end{cases} \tag{15}$$

From (14) and (15) we get

$$p \leqslant 1 \leqslant q. \tag{16}$$

Let

$$\phi_a(x) = f_a(cx) - f_a(x)^2.$$

We have

$$\phi_a(0) = 0 \quad \text{and} \quad \frac{\Gamma(a,0)}{x^{a-1}}\phi_a'(x) = 2e^{-x}f_a(x) - c^a e^{-cx}.$$

This gives: if $2 > c^a$, then ϕ_a attains positive values, and if $2 < c^a$, then ϕ_a attains negative values. Using this result we conclude that if $p > a$, then the first inequality in (14) is not true for all $x > 0$, and if $q < a$, then the second inequality in

(14) does not hold. Thus,

$$p \leqslant a \leqslant q. \tag{17}$$

From (16) and (17) we obtain $p \leqslant \min(a, 1)$ and $q \geqslant \max(a, 1)$. $\square$

Now, we provide bounds for the arithmetic, geometric, and harmonic means of $f_a(x_1), \dots, f_a(x_n)$.

Theorem 2. *Let a be a positive real number. The inequalities*

$$f_a\big(M_r(x_1, \dots, x_n)\big) \leqslant \frac{f_a(x_1) + \dots + f_a(x_n)}{n} \leqslant f_a\big(M_s(x_1, \dots, x_n)\big) \tag{18}$$

hold for all positive real numbers $x_1, \dots, x_n$ $(n \geqslant 2)$ if and only if $r \geqslant a$ and $s = -\infty$.

Proof. Since f_a is strictly decreasing on $[0, \infty)$, we conclude that the right-hand side of (18) with $s = -\infty$ is valid for all $x_1, \dots, x_n > 0$.

The power mean is increasing on $\mathbf{R}$ with respect to its order; see [13, p. 26]. This implies that the function

$$t \mapsto f_a\big(M_t(x_1, \dots, x_n)\big)$$

is decreasing on $\mathbf{R}$, so that it is enough to prove the left-hand side of (18) for $r = a$. Applying Lemma 1 (i) we obtain for $x_1, \dots, x_n > 0$:

$$u_a\left(\frac{x_1^a + \dots + x_n^a}{n}\right) \leqslant \frac{u_a(x_1^a) + \dots + u_a(x_n^a)}{n},$$

which is equivalent to the left-hand side of (18) with $r = a$.

We assume that the first inequality in (18) holds for all $x_1, \dots, x_n > 0$. Then we get for $x, y > 0$:

$$0 \leqslant f_a(x) + (n-1) f_a(y) - n f_a\big(M_r(x, y, \dots, y)\big) = K_{a,r}(x, y), \quad \text{say}.$$

Since

$$K_{a,r}(y, y) = \frac{\partial}{\partial x} K_{a,r}(x, y)\bigg|_{x=y} = 0,$$

we obtain

$$\frac{\partial^2}{\partial x^2} K_{a,r}(x, y)\bigg|_{x=y} = \frac{n-1}{n} \frac{e^{-y} y^{a-2}}{\Gamma(a, 0)} (y + r - a) \geqslant 0.$$

This leads to $r \geqslant a$.

Finally, we suppose that there exists a real number s such that the right-hand side of (18) holds for all $x_1, \dots, x_n > 0$. We consider two cases.

Case 1. $s \geqslant 0$.

If x_1 tends to ∞, then the left-hand side tends to $f_a(x_2) + \dots + f_a(x_n)$, whereas the right-hand side converges to 0. Contradiction!

Case 2. $s < 0$.

We set $x_1 = x$, $x_2 = \dots = x_n = y$, and $c = n^{-1/s}$. If y tends to ∞, then we obtain for $x > 0$:

$$0 \leqslant n f_a(cx) - f_a(x) = \theta_a(x), \quad \text{say}. \tag{19}$$

We have

$$\Gamma(a, 0) e^x x^{1-a} \theta_a'(x) = 1 - n c^a e^{(1-c)x}.$$

Since $c > 1$, there exists a number x^* such that

$$\theta_a'(x) > 0 \quad \text{for } x \geqslant x^*.$$

We have

$$\lim_{x\to\infty}\theta_a(x)=0.$$

It follows that θ_a is negative on $[x^*,\infty)$. This contradicts (19). $\square$

Theorem 3. *Let a be a positive real number. The inequalities*

$$f_a\big(M_u(x_1,\ldots,x_n)\big)\leqslant\big(f_a(x_1)\cdots f_a(x_n)\big)^{1/n}\leqslant f_a\big(M_v(x_1,\ldots,x_n)\big) \tag{20}$$

hold for all positive real numbers $x_1,\ldots,x_n$ ($n\geqslant 2$) if and only if

$$u\geqslant\max(a,1)\quad\textit{and}\quad v\leqslant\min(a,1). \tag{21}$$

Proof. We apply Lemma 1 (ii), (iii). If $0<a<1$, then

$$v_a\left(\frac{x_1+\cdots+x_n}{n}\right)\leqslant\frac{v_a(x_1)+\cdots+v_a(x_n)}{n}=\frac{w_a(x_1^a)+\cdots+w_a(x_n^a)}{n}\leqslant w_a\left(\frac{x_1^a+\cdots+x_n^a}{n}\right). \tag{22}$$

And, if $a\geqslant 1$, then (22) holds with "$\geqslant$" instead of "$\leqslant$". This reveals that (20) is valid with $u\geqslant\max(a,1)$ and $v\leqslant\min(a,1)$.

Next, we show that (20) implies (21). We set $x_1=x$, $x_2=\cdots=x_n=y$. Then the right-hand side of (20) leads to

$$\big(f_a(x)f_a(y)^{n-1}\big)^{1/n}\leqslant f_a\big(M_v(x,y,\ldots,y)\big). \tag{23}$$

We assume that $v>a$ and set $r=n^{-1/v}$. If y tends to 0, then (23) leads to

$$0\leqslant f_a(rx)-f_a(x)^{1/n}=\Delta_{a,r}(x),\quad\text{say}. \tag{24}$$

Differentiation yields

$$\Delta'_{a,r}(x)=\frac{x^{a-1}}{\Gamma(a,0)}\eta_{a,r}(x), \tag{25}$$

where

$$\eta_{a,r}(x)=\frac{1}{n}e^{-x}f_a(x)^{1/n-1}-r^ae^{-rx}.$$

Since $v>a$, we get

$$\lim_{x\to 0}\eta_{a,r}(x)=\frac{1}{n}-r^a<0. \tag{26}$$

From (25) and (26) we conclude that $\Delta_{a,r}$ is strictly decreasing in the neighbourhood of 0. This contradicts (24), since $\Delta_{a,r}(0)=0$. Thus, $v\leqslant a$. Now, we assume that $v>1$. From (23) we obtain

$$\frac{\Gamma(a,x)}{e^{-x}x^{a-1}}\Gamma(a,y)^{n-1}\leqslant I_a(x)\left(\frac{\Gamma(a,\chi)}{e^{-\chi}\chi^{a-1}}\right)^n \tag{27}$$

with

$$I_a(x)=\frac{(e^{-\chi}\chi^{a-1})^n}{e^{-x}x^{a-1}}\quad\text{and}\quad\chi=\left(\frac{x^v+(n-1)y^v}{n}\right)^{1/v}.$$

We have

$$\frac{\log I_a(x)}{x}=1+(a-1)\left(n\frac{\log\chi}{x}-\frac{\log x}{x}\right)-n\frac{\chi}{x}.$$

Since

$$\lim_{x\to\infty}\frac{\log\chi}{x}=0\quad\text{and}\quad\lim_{x\to\infty}\frac{\chi}{x}=n^{-1/v},$$

we get

$$\lim_{x\to\infty}\frac{\log I_a(x)}{x}=1-n^{1-1/v}<0.$$

Hence,

$$\lim_{x\to\infty} I_a(x) = 0. \tag{28}$$

Applying (28) and

$$\lim_{x\to\infty} \frac{\Gamma(a,x)}{e^{-x}x^{a-1}} = 1 \tag{29}$$

(see [1, p. 263]), we obtain from (27): $\Gamma(a,y)^{n-1} \leqslant 0$. A contradiction! Thus, $v \leqslant 1$.

If the left-hand side of (20) holds, then we get $u \geqslant \min(a,1)$. Therefore, $u > 0$. We assume that $u < a$, set $x_1 = x$, $x_2 = \cdots = x_n = y$, and let y tend to 0. Then we obtain

$$\Delta_{a,s}(x) < 0 \quad \text{with } s = n^{-1/u}.$$

We have

$$\Delta'_{a,s}(x) = \frac{x^{a-1}}{\Gamma(a,0)}\eta_{a,s}(x) \quad \text{and} \quad \lim_{x\to 0} \eta_{a,s}(x) = \frac{1}{n} - s^a > 0.$$

This gives $\Delta_{a,s}(x) > \Delta_{a,s}(0) = 0$ for all sufficiently small x. A contradiction! Hence, $u \geqslant a$. Next, we suppose that $u < 1$. Again, we set $x_1 = x$, $x_2 = \cdots = x_n = y$. Then we get

$$\left(\frac{\Gamma(a,\rho)}{e^{-\rho}\rho^{a-1}}\right)^n \leqslant \frac{\Gamma(a,x)}{e^{-x}x^{a-1}}\Gamma(a,y)^{n-1} J_a(x) \tag{30}$$

with

$$J_a(x) = \frac{e^{-x}x^{a-1}}{(e^{-\rho}\rho^{a-1})^n} \quad \text{and} \quad \rho = \left(\frac{x^u + (n-1)y^u}{n}\right)^{1/u}.$$

Since

$$\lim_{x\to\infty} \frac{\Gamma(a,\rho)}{e^{-\rho}\rho^{a-1}} = \lim_{x\to\infty} \frac{\Gamma(a,x)}{e^{-x}x^{a-1}} = 1 \quad \text{and} \quad \lim_{x\to\infty} J_a(x) = 0,$$

we obtain from (30): $1 \leqslant 0$. This contradiction leads to $u \geqslant 1$. $\square$

Our next theorem presents a double-inequality for the harmonic mean of $f_a(x_1), \ldots, f_a(x_n)$. We only settle the case $a \in (0,1]$ completely. It remains an open problem to determine all parameters β such that the right-hand side of (31) (given below) is valid in the case of $a > 1$.

Theorem 4. *Let $0 < a \leqslant 1$. The inequalities*

$$f_a\big(M_\alpha(x_1,\ldots,x_n)\big) \leqslant \frac{n}{1/f_a(x_1) + \cdots + 1/f_a(x_n)} \leqslant f_a\big(M_\beta(x_1,\ldots,x_n)\big) \tag{31}$$

hold for all positive real numbers $x_1, \ldots, x_n$ ($n \geqslant 2$) if and only if $\alpha = \infty$ and $\beta \leqslant a$. Moreover, when $a > 1$, then the left-hand side of (31) *holds if and only if $\alpha = \infty$.*

Proof. Since $1/f_a$ is increasing on $(0,\infty)$, we obtain

$$\frac{1}{n}\left(\frac{1}{f_a(x_1)} + \cdots + \frac{1}{f_a(x_1)}\right) \leqslant \max_{1\leqslant i\leqslant n} \frac{1}{f_a(x_i)} = \frac{1}{f_a(\max_{1\leqslant i\leqslant n} x_i)} = \frac{1}{f_a(M_\infty(x_1,\ldots,x_n))}.$$

Next, we assume that there exists a real number α such that (31) holds for all $x_1, \ldots, x_n > 0$. Applying the geometric mean – harmonic mean inequality and Theorem 3 gives

$$f_a\big(M_\alpha(x_1,\ldots,x_n)\big) \leqslant f_a\big(M_v(x_1,\ldots,x_n)\big) \quad \text{with } v = \min(a,1).$$

This implies $\alpha \geqslant v > 0$. We set $x_1 = x$, $x_2 = \cdots = x_n = y$ and let y tend to 0. Then we obtain from (31):

$$(n-1)f_a(bx) + \frac{f_a(bx)}{f_a(x)} \leqslant n \quad \text{with } b = n^{-1/\alpha}. \tag{32}$$

Since

$$\lim_{x\to\infty} \frac{f_a(\lambda x)}{f_a(x)} = \lim_{x\to\infty} \lambda^a e^{(1-\lambda)x} = \begin{cases} \infty, & \text{if } 0 < \lambda < 1, \\ 0, & \text{if } \lambda > 1, \end{cases}$$

we conclude from (32) that $1 \leqslant b = n^{-1/\alpha}$. This contradicts $\alpha > 0$.

Applying Lemma 1 (ii) implies that if $a \in (0,1)$, then $x \mapsto f_a(x^{1/a})^{-1}$ is log-convex on $[0,\infty)$. This is also true, if $a=1$. It follows that $x \mapsto f_a(x^{1/a})^{-1}$ is convex, so that we obtain

$$f_a(z)^{-1} \leqslant \frac{f_a(x_1)^{-1} + \cdots + f_a(x_n)^{-1}}{n} \quad \text{with } z = \left(\frac{x_1^a + \cdots + x_n^a}{n}\right)^{1/a}.$$

This leads to the right-hand side of (31) with $\beta = a$.

We assume that there exists a number $\beta > a$ such that (31) is valid for all $x_1, \dots, x_n > 0$. Then we set $x_1 = x$, $x_2 = \cdots = x_n = y$ and let y tend to 0. This yields

$$\sigma_a(x) = f_a(cx) + (n-1) f_a(x) f_a(cx) - n f_a(x) \geqslant 0 = \sigma_a(0) \quad \text{with } c = n^{-1/\beta}. \tag{33}$$

Differentiation gives

$$\frac{\Gamma(a,0)}{x^{a-1}} \sigma_a'(x) = -c^a e^{-cx} - (n-1)\left[e^{-x} f_a(cx) + c^a e^{-cx} f_a(x)\right] + n e^{-x}.$$

Since

$$\lim_{x\to 0} \frac{\Gamma(a,0)}{x^{a-1}} \sigma_a'(x) = 1 - nc^a < 0,$$

we conclude that σ_a attains negative values. This contradicts (33). Thus, $\beta \leqslant a$. $\square$

From Lemma 1 (iv) we obtain

$$f_a(x)^\lambda f_a(y)^{1-\lambda} < f_a\left(x^\lambda y^{1-\lambda}\right) \quad \text{for } a > 0,\ x, y > 0\ (x \neq y),\ \lambda \in (0,1). \tag{34}$$

In the final part of this section, we prove that for every $x > 0$ the function $a \mapsto \log f_a(x)$ is strictly concave on $(0,\infty)$. This result leads to a companion of (34).

Theorem 5. *The inequality*

$$f_a(x)^\lambda f_b(x)^{1-\lambda} < f_{\lambda a + (1-\lambda) b}(x)$$

is valid for all $a, b > 0$ $(a \neq b)$, $x > 0$, and $\lambda \in (0,1)$. In particular, the Turán-type inequality

$$f_a(x) f_{a+2}(x) < \left[f_{a+1}(x)\right]^2$$

holds for all $a > 0$ and $x > 0$.

Proof. We show that

$$\frac{\partial^2}{\partial a^2} \log f_a(x) < 0 \tag{35}$$

for $a > 0$ and $x > 0$. Let $\psi = \Gamma'/\Gamma$ and $\Gamma(a) = \Gamma(a,0)$. Then we have

$$\begin{aligned}\Gamma(a,x)^2 \frac{\partial^2}{\partial a^2} \log f_a(x) = {} & \int_x^\infty e^{-t} t^{a-1}\,dt \int_x^\infty e^{-t} t^{a-1} (\log t)^2\,dt \\ & - \left(\int_x^\infty e^{-t} t^{a-1} \log t\,dt\right)^2 - \psi'(a) \left(\int_x^\infty e^{-t} t^{a-1}\,dt\right)^2.\end{aligned} \tag{36}$$

We denote the expression on the right-hand side of (36) by $U_a(x)$. Then we get

$$\begin{aligned}e^x x^{1-a} \frac{\partial}{\partial x} U_a(x) &= -(\log x)^2 \int_x^\infty e^{-t} t^{a-1}\,dt - \int_x^\infty e^{-t} t^{a-1} (\log t)^2\,dt \\ &\quad + 2(\log x) \int_x^\infty e^{-t} t^{a-1} \log t\,dt + 2\psi'(a) \int_x^\infty e^{-t} t^{a-1}\,dt \\ &= V_a(x), \quad \text{say}.\end{aligned} \tag{37}$$

We differentiate $V_a(x)$ with respect to x and obtain

$$\frac{e^x x^{1-a}}{2}\frac{\partial}{\partial x}V_a(x) = e^x\int_1^\infty e^{-xt}t^{a-1}\log t\,dt - \psi'(a) = W_a(x), \quad \text{say.} \tag{38}$$

Using $\log t \leqslant t-1$ for $t \geqslant 1$, we find

$$0 < e^x\int_1^\infty e^{-xt}t^{a-1}\log t\,dt \leqslant e^x\left(\int_1^\infty e^{-xt}t^a\,dt - \int_1^\infty e^{-xt}t^{a-1}\,dt\right) = \frac{1}{x}\left(\frac{\Gamma(a+1,x)}{e^{-x}x^a} - \frac{\Gamma(a,x)}{e^{-x}x^{a-1}}\right). \tag{39}$$

From (29), (38), and (39) we conclude that

$$\lim_{x\to\infty} W_a(x) = -\psi'(a) < 0.$$

Moreover, we have

$$\frac{\partial}{\partial x}W_a(x) = e^x\int_1^\infty e^{-xt}(1-t)t^{a-1}\log t\,dt < 0.$$

We assume that W_a attains only negative values on $(0,\infty)$. Then, (38) implies that V_a is strictly decreasing on $(0,\infty)$. From (37) we obtain

$$\lim_{x\to 0}\frac{V_a(x)}{(\log x)^2} = -\Gamma(a,0) \quad \text{and} \quad \lim_{x\to 0} V_a(x) = -\infty. \tag{40}$$

A contradiction! This implies that there exists a positive number $\tilde{x}$ such that W_a is positive on $(0,\tilde{x})$ and negative on $(\tilde{x},\infty)$. Using (38) gives that V_a is strictly increasing on $(0,\tilde{x}]$ and strictly decreasing on $[\tilde{x},\infty)$. Hospital's rule leads to

$$\lim_{x\to\infty} V_a(x) = 0. \tag{41}$$

From (40), (41), and the monotonicity of V_a we obtain that there exists a positive number $\hat{x}$ such that V_a is negative on $(0,\hat{x})$ and positive on $(\hat{x},\infty)$. Applying (37) yields that U_a is strictly decreasing on $(0,\hat{x}]$ and strictly increasing on $[\hat{x},\infty)$. We have

$$U_a(0) = \lim_{x\to\infty} U_a(x) = 0.$$

Thus, $U_a(x) < 0$ for $x > 0$. This proves (35). $\square$

Remark. Further Turán-type inequalities for special functions are given in [9].

3. Additional results and remarks

(I) Applying Lemmas 1 (i), 2, and the monotonicity of f_a we obtain the following sharp inequalities. *Let $a > 0$ be a real number and $n \geqslant 2$ be an integer. For all $x_1,\ldots,x_n \geqslant 0$ we have*

$$0 < f_a\big(x_1^{1/a}\big) + \cdots + f_a\big(x_n^{1/a}\big) - f_a\big((x_1+\cdots+x_n)^{1/a}\big) \leqslant n-1.$$

Both bounds are best possible.

(II) *Let $a > 0$, $b \neq 0$, and $c \neq 0$ be real numbers. The function $x \mapsto [f_a(x^b)]^c$ is strictly subadditive on $[0,\infty)$, that is,*

$$\big[f_a\big((x+y)^b\big)\big]^c < \big[f_a\big(x^b\big)\big]^c + \big[f_a\big(y^b\big)\big]^c \quad \text{for all } x, y \geqslant 0, \tag{42}$$

if and only if $bc > 0$.

Let $x > 0$. If $bc > 0$, then we have

$$\frac{d}{dx}\big[f_a\big(x^b\big)\big]^c = -\frac{bc}{\Gamma(a,0)}x^{ab-1}e^{-x^b}\big[f_a\big(x^b\big)\big]^{c-1} < 0.$$

This leads to (42). Conversely, if (42) holds, then we obtain

$$\left(\frac{f_a(2^b x^b)}{f_a(x^b)}\right)^c < 2. \tag{43}$$

Case 1. $b > 0$. We have

$$\lim_{x\to\infty} \frac{f_a(2^b x^b)}{f_a(x^b)} = 0. \tag{44}$$

From (43) and (44) we get $c > 0$.

Case 2. $b < 0$. Then

$$\lim_{x\to 0} \frac{f_a(2^b x^b)}{f_a(x^b)} = \infty, \tag{45}$$

so that (43) and (45) lead to $c < 0$.

(III) *Let $a > 0$ be a real number. The inequality*

$$f_a(x) + f_a(y) \leqslant 1 + f_a(z) \tag{46}$$

holds for all nonnegative real numbers x, y, z with $x^2 + y^2 = z^2$ if and only if $a \leqslant 2$.

To prove (46) for $a \in (0, 2]$ we define

$$\Omega_a(x, y) = 1 + f_a\big(\sqrt{x^2 + y^2}\big) - f_a(x) - f_a(y) \quad \text{and} \quad \omega_a(t) = -\frac{e^{-t} t^{a-2}}{\Gamma(a, 0)}.$$

Partial differentiation gives

$$\frac{\partial}{\partial x} \Omega_a(x, y) = x\big(\omega_a\big(\sqrt{x^2 + y^2}\big) - \omega_a(x)\big).$$

Since

$$\omega_a'(t) = \frac{e^{-t} t^{a-3}}{\Gamma(a, 0)} (t + 2 - a) > 0 \quad \text{for } t > 0,$$

we conclude that $x \mapsto \Omega_a(x, y)$ is strictly increasing on $[0, \infty)$. This leads to

$$\Omega_a(x, y) \geqslant \Omega_a(0, y) = 0.$$

Conversely, if (46) is valid with $a > 2$, then we get for $x, y \geqslant 0$:

$$\Omega_a(x, y) \geqslant 0 = \Omega_a(0, y). \tag{47}$$

We have

$$\frac{\partial}{\partial x} \Omega_a(x, y)\bigg|_{x=0} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2} \Omega_a(x, y)\bigg|_{x=0} = \omega_a(y) < 0.$$

This contradicts (47).

Inequality (46) is a Grünbaum-type inequality; see [12,7].

(IV) Lemma 1 leads to the next result: *For all nonnegative real numbers x, y, z with $x \leqslant z$ we have*

$$f_a\big((x + y)^{1/a}\big) + f_a\big(z^{1/a}\big) \leqslant f_a\big(x^{1/a}\big) + f_a\big((y + z)^{1/a}\big) \quad (a > 0)$$

and

$$f_a\big((x + y)^{1/a}\big) \cdot f_a\big(z^{1/a}\big) \leqslant f_a\big(x^{1/a}\big) \cdot f_a\big((y + z)^{1/a}\big) \quad (a \geqslant 1). \tag{48}$$

If $0 < a < 1$, then (48) *holds with "$\geqslant$" instead of "$\leqslant$".*

(V) A function $h : [0, \infty) \to \mathbf{R}$ is called completely monotonic, if h is continuous on $[0, \infty)$ and satisfies

$$(-1)^n h^{(n)}(x) \geqslant 0 \quad (x > 0,\ n = 0, 1, 2, \ldots).$$

Detailed information on these functions can be found in [3,4]. Let $0 < a \leqslant 1$. The representation

$$f_a(x) = \frac{e^{-x}}{\Gamma(a, 0)} \int_0^\infty e^{-t} (t + x)^{a-1}\, dt$$

reveals that f_a can be written as a product of two completely monotonic functions. This implies that f_a is completely monotonic. Conversely, if f_a is completely monotonic, then we conclude from

$$0 \leqslant x^{2-a} e^x \Gamma(a,0) f_a''(x) = x - (a-1)$$

that $a \leqslant 1$. Hence, *f_a is completely monotonic on $[0,\infty)$ if and only if $0 < a \leqslant 1$.*

We have

$$\frac{1-f_a(x)}{x^a} = \frac{1}{\Gamma(a,0)} \int_0^1 e^{-xt} t^{a-1}\,dt.$$

Thus, *$x \mapsto (1-f_a(x))/x^a$ $(a>0)$ is completely monotonic on $[0,\infty)$.* See [18].

(VI) Kimberling [15] proved that if $h : [0,\infty) \to (0,1]$ is completely monotonic, then

$$h(x)h(y) \leqslant h(x+y) \quad (x, y \geqslant 0).$$

Since $0 < f_a(x) \leqslant 1$ for $x \geqslant 0$, we conclude that the second inequality in (1) holds. See also [8].

If $a \geqslant 1$, then $0 < (1-f_a(x))/x^a \leqslant 1$ for $x \geqslant 0$. This leads to

$$\frac{(1-f_a(x))(1-f_a(y))}{1-f_a(x+y)} \leqslant \left(\frac{xy}{x+y}\right)^a \quad (a \geqslant 1,\ x, y > 0).$$

(VII) The following interesting upper bound for $f_a(x)$ was discovered by Laforgia and Natalini [16]:

$$f_a(x) < 1 + \frac{x^a}{\Gamma(a,0)} \left(\frac{1}{a+1} \int_0^x \frac{1-e^{-t}}{t}\,dt - \frac{1}{a} \right) \quad (0 < a < 1,\ x > 0). \tag{49}$$

Here, we offer a short and simple new proof, which reveals that (49) is also valid for $a \geqslant 1$. We define for $a, x > 0$ and $p, y > 0$:

$$I_a(x) = 1 + \frac{x^a}{\Gamma(a,0)} \left(\frac{1}{a+1} \int_0^x \frac{1-e^{-t}}{t}\,dt - \frac{1}{a} \right) - f_a(x),$$

$$J_p(y) = \int_0^y e^{-t^p}\,dt + y\left(\frac{p}{p+1} \int_0^y \frac{1-e^{-t^p}}{t}\,dt - 1 \right).$$

Since $J_p(0) = J_p'(0) = 0$ and

$$J_p''(y) = \frac{pe^{-y^p}}{(p+1)y}\left(e^{y^p} - 1 - y^p\right) > 0,$$

we conclude that $J_p(y) > 0$. The identity

$$\Gamma(a+1,0) I_a(x) = J_{1/a}\left(x^a\right)$$

reveals that $I_a(x) > 0$.

(VIII) Let $g : [0,\infty) \to [0,\infty)$ be a probability density function and $G, \bar{G} : [0,\infty) \to (0,1]$, defined by

$$G(x) = \int_0^x g(t)\,dt, \qquad \bar{G}(x) = 1 - g(x) = \int_x^\infty g(t)\,dt$$

be the corresponding cumulative distribution function and complementary cumulative distribution function (sometimes called as reliability or survival function), respectively. By definition, a life distribution (with cumulative distribution function G such that $G(x) = 0$ for all $x < 0$) has the increasing failure rate (IFR) property if $x \mapsto g(x)/\bar{G}(x) = -\bar{G}'(x)/\bar{G}(x)$ is increasing on $[0,\infty)$, that is, the reliability function $\bar{G}$ is log-concave. It is well known that if a probability density function is log-concave, then the corresponding cumulative distribution function and the complementary cumulative distribution function have the same property (for more details see [5,6,8]). Another class of life distributions is the NBU, which has been shown to be fundamental in the study of replacement policies. By definition, a life distribution satisfies the new-is-better-than-used (NBU) property if $x \mapsto \log \bar{G}(x)$ is subadditive, that is,

$$\bar{G}(x+y) \leqslant \bar{G}(x)\bar{G}(y)$$

for all $x, y \geqslant 0$. The corresponding concept of a new-is-worse-than-used (NWU) distribution is defined by reversing the above inequality. We note that the NBU property may be interpreted as stating that the chance $\bar{G}(x)$ that a new unit will survive to age x is greater than the chance $\bar{G}(x+y)/\bar{G}(y)$ that a survived unit of age y will survive for an additional time x. It can be shown that if a life distribution is IFR, then it is NBU (see, for example, [8]), but the inverse implication in general does not hold.

The function f_a is actually the survival function of the gamma distribution. More precisely, the gamma function has support $[0, \infty)$, probability density function and reliability function

$$x \mapsto \frac{e^{-x}x^{a-1}}{\Gamma(a,0)} \quad \text{and} \quad x \mapsto f_a(x),$$

where $a > 0$ is the shape parameter, which is the mean of a gamma-distributed random variable. Taking into account the above observation, recently, it was pointed out in [8] that the first inequality in (1) is the NBU property for the gamma distribution, while the second inequality in (1) is the NWU property for the gamma distribution.

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